Closed form analytical inverse solutions for Risley-prism-based beam steering systems in different configurations

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Received 20 April 2011; accepted 30 May 2011; posted 16 June 2011 (Doc. ID 146180); published 21 July 2011

Nonparaxial ray tracing through Risley prisms of four different configurations is performed to give the exact solution of the inverse problem arisen from applications of Risley prisms to free space communications. Predictions of the exact solution and the third-order theory [Appl. Opt. 50, 679 (2011)] are compared and results are shown by curves for systems using prisms of different materials. The exact solution for the problem of precision pointing is generalized to investigate the synthesis of the scan pattern, i.e., to create a desirable scan pattern on some plane perpendicular to the optical axis of the system by controlling the circular motion of the two prisms.© 2011 Optical Society of America

1. Introduction

Risley prisms are pairs of rotatable prisms that can be used to continuously scan a laser beam over a wide angular range with a high resolution. There are two basic problems encountered in the applications of Risley-prism-based systems to laser beam steering. First, given the two prisms’ orientations, what is the pointing position of the laser beam emergent from the system? The second problem may be regarded as the inverse of the first problem, i.e., given the required pointing position, what will be the orientations of two prisms? The answer to the first problem can be found in a number of publications [1–4]. Investigations of the second problem, which is known as the inverse problem or the problem of precision pointing [4–8], have been stimulated by the applications of Risley prisms to optical tracking and pointing of targets in free space. Solutions for the inverse problem, which can be regarded as exact, are very rare in literature. Historically, in the 1985 paper of Amirault and DiMarzio [5], they found difficulties in inverting the vector equations for refraction at the surfaces of the prisms for an exact solution to the problem of precise pointing using Risley prisms. However, they proposed a realistic approach to obtain an inverse solution through a two-step process.

Ten years later in 1995, Boisset et al. [6] proposed a paraxial method and developed an iterative algorithm to solve the inverse problem. Furthermore, Degnan [7] developed in 2002 a first-order approximation method that does not need an iterative algorithm to find prisms’ orientations for a given pointing position. Recently, Li proposed a third-order solution for the inverse problem [4]. The first exact solution was given in 2008 by Yang [8], who investigated the case of Risley prisms having two identical dispersive prisms, i.e., the prisms having the isosceles triangular cross sections, whereas this study presents exact inverse solutions for systems containing two wedge prisms on account of the wedge prisms being ideal for laser beam steering and are now commonly used in the real devices.

Figure 1 is a schematic diagram illustrating the configuration of Risley prisms, in which the two wedge prisms $\Pi_1$ and $\Pi_2$ may have different indexes $n_1$ and $n_2$ and different opening angles $a_1$ and $a_2$. The principal cross sections of $\Pi_1$ and $\Pi_2$ are right...
triangles. If the opening angle of the prism is taken as a reference to define the three sides of a right triangle and if symbol 1 is used to represent the side adjacent to the opening angle and symbol 2 for the hypotenuse, the configuration shown in Fig. 1 can therefore be described by a combination of the symbols as 21-12. Making use of this symbology, we may have a system in the 12-12 configuration if the first prism \( \Pi_1 \) in Fig. 1 is turned 180° around the y axis. In general, there are four different configurations of the prism systems shown in Figs. 1(a)–1(d): 21-12, 12-21, 21-21, and 12-21. Different configurations may have different assumptions: no prism tilt and no beam and mechanical axis misalignment will be considered throughout this study.

The outline of this study is as follows. In Section 2, nonparaxial ray tracing through the prism systems in different configurations is performed under the basis of the more exact thick prism theory. An exact solution to the problem of precision pointing is developed in Section 3. Predictions of the exact solution and the third-order theory are compared in Section 4. Generalization of the exact result for the problem of precision pointing to scan pattern synthesis is investigated in Section 5. Conclusions are drawn at the end of this study.

This study is conducted under the error-free assumptions: no prism tilt and no beam and mechanical axes misalignment will be considered throughout this study.

2. Nonparaxial Ray Tracing

Having illustrated the two prisms system in different configurations in Figs. 1 and 2, we can shine a ray through the systems shown in Fig. 2 to compare their powers of ray deviation.

The two prisms shown in Fig. 1 are rotatable about the z axis, counted from the xz plane. Their angular positions are specified by their respective rotation angles \( \theta_1 \) and \( \theta_2 \). The incident ray propagates in the direction specified by the ray vector \( \mathbf{s}_1^{(i)} = (0, 0, -1) \) and hits the center of the first surface of prism \( \Pi_1 \). In case of the 21-12 configuration and when \( \Pi_1 \) is in the position of \( \theta_1 \), the first surface of \( \Pi_1 \) is now in the position specified by the unit normal vector

\[
\mathbf{n}_1 = (\sin \alpha_1 \cos \theta_1, \sin \alpha_1 \sin \theta_1, \cos \alpha_1).
\]  

Apply the vector form Snell's law \([9]\) to \( \Pi_1 \); the incident ray is refracted at the first surface of \( \Pi_1 \), separating air and glass, into the direction specified by the ray vector

\[
\mathbf{s}_1^{(r)} = \frac{1}{n_1} [\mathbf{s}_1^{(i)} - (\mathbf{s}_1^{(i)} \cdot \mathbf{n}_1) \mathbf{n}_1].
\]

In case of the 21-12 configuration and when \( \Pi_1 \) is in the position of \( \theta_1 \), the first surface of \( \Pi_1 \) is in the position specified by the unit normal vector

\[
\mathbf{n}_2 = (-\sin \alpha_2 \cos \theta_2, -\sin \alpha_2 \sin \theta_2, \cos \alpha_2).
\]

Again, apply the vector form Snell's law to refraction calculations for the ray emergent from the system and we obtain the ray vector

\[
\mathbf{s}_2^{(r)} = n_2 [\mathbf{s}_1^{(r)} - (\mathbf{s}_1^{(r)} \cdot \mathbf{n}_2) \mathbf{n}_2] - (\mathbf{n}_2) \sqrt{1 - n_2^2 + n_2^2 (\mathbf{s}_1^{(r)} \cdot \mathbf{n}_2)^2}.
\]

After a long mathematical development, omitted here, we can write expressions for the direction cosines \( (K_A, L_A, M_A) \) of the ray vector \( \mathbf{s}_2^{(r)} \) for the ray emergent from the system in the 21-12 configuration in the form

\[
K_A = \alpha_1 \cos \theta_1 + \alpha_2 \sin \alpha_2 \cos \theta_2.
\]
\[ L_A = a_1 \sin \theta_1 + a_3 \sin \alpha_2 \sin \theta_2 \]  
and  
\[ M_A = a_2 - a_3 \cos \alpha_2. \]

Expressions for the coefficients \((a_1, a_2, a_3)\) are shown in Eqs. (T1.1) to (T1.5) in Table 1.

The next step is to perform the nonparaxial ray tracing for the Risley prisms in the 12-12 configuration as shown in Fig. 2(b). Interestingly, we found that the direction cosines of the ray vector for the ray emergent from the system in the 12-12 configuration can be expressed in the same forms as Eqs. (2.5a)–(2.5c) for the system in the 21-12 configuration. This finding can be explained as follows.

The first prism \(\Pi_1\) in the two configurations is in different arrangement, i.e., the 21 and 12 arrangements, and, according to the more exact thick prisms theory, these two prisms may have different power of ray deviation, say, \(\delta_{21}\) and \(\delta_{12}\), and \(\delta_{21} \neq \delta_{12}\). As \(\Pi_1\) is rotated about the \(z\) axis, the ray deviated by \(\Pi_1\) will trace out a circular cone symmetrical to the \(z\) axis. The half-vertex angles of the circular cones are \(\delta_{21}\) and \(\delta_{12}\) for the two arrangements, respectively. However, they are all homocentric pencils and symmetrically distributed around the optical axes of the systems. The point of great importance is that the second prism \(\Pi_2\) in the two systems is in the same arrangement and the incident rays to \(\Pi_2\) are uniformly distributed in circular cone beams coming out from \(\Pi_1\), whereas the deviation of the half-vertex \(\delta_{21}\) and \(\delta_{12}\) of the cones may experience some qualitative and quantitative differences in the numerical outcomes of the scan field but may not be able to change the mathematical formulism of the scan field. This is why systems in the 21-12 and 12-12 configurations share the same expressions as shown in Eqs. (2.5a)–(2.5c). Therefore, we may consider the two systems in the 21-12 and 12-12 configurations as two members in one group, as shown by the group A in Fig. 2.

Similarly, the prisms systems in the 21-21 and 21-12 configurations as shown in Figs. 2(c) and 2(d) can be considered as the two members in the group B (see Fig. 2). Because the direction cosines \((K_B, L_B, M_B)\) of the rays emerging from these two systems can be expressed in the same forms as

\[ K_B = b_1 \cos \theta_1 - b_3 \sin \alpha_2 \cos \theta_2, \]  
\[ L_B = b_1 \sin \theta_1 - b_3 \sin \alpha_2 \sin \theta_2, \]  
and  
\[ M_B = -\sqrt{1 - n_3^2 + (b_2 + b_3 \cos \alpha_2)^2}. \]

Expressions for the coefficients \((b_1, b_2, b_3)\) are shown in Eqs. (T2.1) to (T2.5) in Table 2.

The next step is a comparison of the power of ray deviation of systems in different configurations. It is known that a pair of rotatable prisms is equivalent to a single prism of variable power and the maximum ray deviation angle \(\Phi_m\) is obtained when the prism apexes are aligned \([1,2]\). Variations of \(\Phi_m\) are plotted from Eqs. (2.5) and (2.6) as a function of prism opening angle. Results are shown by curves in Fig. 3(a) for systems having two identical prisms of opening angle \(\alpha_1 = \alpha_2 = \alpha\) and indices \(n_1 = n_2 = n = 1.5\) (glass prisms) and \(n_1 = n_2 = n = 4.0\) (silicon prisms) by the curves in Fig. 3(b). All the curves in Figs. 3(a) and 3(b) have a maximum when the prism opening angle is small. More specifically, the results show in Fig. 3(a) that the difference between the predictions of the exact solution and the third-order theory is \(<1\%\) when \(\alpha < 3^\circ\) and the group A in 3(b) shows the difference \(<0.03\%\) when \(\alpha < 5^\circ\). After the linear range, the difference increases to 6.4\% when \(\alpha = 30^\circ\) and to 0.2\% when \(\alpha = 8^\circ\) in the cases shown in Figs. 3(a) and 3(b), respectively.

Finally, it is interesting to mention that the systems in group A may have greater power of ray deviation than the systems in the group B beyond the linear range
the linear variation range shown in Figs. 3(a) and 3(b).

3. Inverse Solutions for Risley Prisms of Different Configurations
Attention is now turned to the exact solutions of the problem of precision pointing, i.e., the problem of steering a laser beam to any specific altitude $\Phi$ and azimuth $\Theta$ within the angular range of the system [2,3]. Here $\Phi$ is the angle of the beam relative to the $z$ axis and $\Theta$ is the angle around the $z$ axis, counted from the $x\text{-}z$ plane (see Fig. 1).

To steer the beam to the direction specified by $(\Phi, \Theta)$, we follow the two-step method proposed by Amirault and DiMarzio [5]. The first step is to keep the prism $\Pi_1$ stationary at $\theta_1 = 0$ and rotate the second prism $\Pi_2$ until the desired altitude $\Phi$ is achieved, i.e., a rotation of $\Pi_2$ relative to $\Pi_1$ until the following equation is satisfied:

$$
\cos \Phi = -\frac{M_B}{\tan \theta_2} \left( a_2 + \frac{1}{\cos \Phi} \right)
$$

(For system in the Group A).

$$
\cos \Phi = -M_B \sqrt{1 - n_2^2 + (b_2 + b_3 \cos \alpha_2)^2}
$$

(For system in the Group B).

Upon substituting from Eqs. (T1.3) and (T1.5) in Table 1 for the coefficient $a_2$ and $a_3$ into Eq. (3.1a), we may express the angle of azimuthal rotation between $\Pi_1$ and $\Pi_2$ as

$$
(\Delta \theta)_0 = \arccos \left( \frac{1}{a_1 \tan \alpha_2} \left\{ \frac{a_2}{2(a_2 + \cos \Phi)} \frac{1}{\cos \alpha_2} \right\} \right)
$$

(3.2a)

Similarly, for the prism systems in the group $B$, the angle $(\Delta \theta)_0$ can be obtained by substituting from Eqs. (T2.3) and (T2.5) in Table 2 for $b_2$ and $b_3$ into Eq. (7b). The result is

$$
(\Delta \theta)_0 = \arccos \left( \frac{1}{a_1 \tan \alpha_2} \left\{ \frac{a_2}{2(a_2 + \cos \Phi)} \frac{1}{\cos \alpha_2} \right\} \right)
$$

(3.2b)
\[
\Delta \theta_0 = \arccos \left( \frac{1}{b_1 \tan \alpha_2} \right) \\
\times \left\{ b_2 + \frac{1}{-b_2 \pm \sqrt{\cos^2 \Phi - 1 + n_2^2}} \right. \\
\left. \times \left[ 1 - n_2^2 \cos \alpha_2 \left( -b_2 \pm \sqrt{\cos^2 \Phi - 1 + n_2^2} \right)^2 \right] \right\}. \tag{3.2b}
\]

After the desired altitude is achieved, the beam is pointing to a new direction specified by the angles \((\Phi, \psi_0)\). For systems in the group \(A\), the azimuth displacement \(\psi_0\) introduced in the first step is given by

\[
\psi_0 = \arctan \left( \frac{L_A}{K_A} \right) \theta_1 = 0, \quad \theta_2 = (\Delta \theta)_0 \\
= \arctan \left( \frac{\tan \alpha_2(a_2 + \cos \Phi) \sin(\Delta \theta)_0}{a_1 + \tan \alpha_2(a_2 + \cos \Phi) \cos(\Delta \theta)_0} \right). \tag{3.3a}
\]

Similarly, for prism systems in group \(B\), we have

\[
\psi_0 = \arctan \left( \frac{L_B}{K_B} \right) \theta_1 = 0, \quad \theta_2 = (\Delta \theta)_0 \\
= \arctan \left( \frac{\tan \alpha_2(a_2 + \cos \Phi) \sin(\Delta \theta)_0}{a_1 + \tan \alpha_2(a_2 + \cos \Phi) \cos(\Delta \theta)_0} \right). \tag{3.3b}
\]

The second step in the two-step process is a simultaneous rotation of the prisms \(\Pi_1\) and \(\Pi_2\) about the axis of the system until the desired azimuth \(\Theta\) is reached. The final rotation angles of \(\Pi_1\) and \(\Pi_2\) are given, respectively, by

\[
\theta_1 = \Theta - \psi_0 \quad \text{and} \quad \theta_2 = \theta_1 + (\Delta \theta)_0 = \Theta - \psi_0 + (\Delta \theta)_0. \tag{3.4}
\]

4. Comparison of the Predictions of Different Theories

This section is devoted to a comparison of the exact solution with the predictions of the third-order theory \([4]\). Results in Sections 4 and 5 will be obtained for the system in the 21-12 configuration. However, they can be easily obtained for systems in other configurations if the appropriate formulas in Tables 1 and 2 are used to replace Eqs. (T1.1), (T1.3), and (T1.5) in the calculations of direction cosines of the ray emergent from the system.

Assume \(\delta_1\) and \(\delta_2\) represent the difference between prism rotation angles \(\theta_1\) and \(\theta_2\) predicted by the exact solution and the third-order theory for the problem of precision pointing of a beam to the direction specified by \((\Phi, \Theta)\). Hence we can write

\[
\delta_1 = (\theta_1)_{\text{Exact}} - (\theta_1)_{3\text{rd order}} \quad \text{and} \quad \delta_2 = (\theta_2)_{\text{Exact}} - (\theta_2)_{3\text{rd order}}. \tag{4.1}
\]

Upon substitution of Eq. (3.4) in this study and Eq. (3.9) in [4] into Eq. (4.1), we can further express Eq. (4.1) in the form

\[
\delta_1 = -(\psi_0)_{\text{Exact}} + (\psi_0)_{3\text{rd order}} \quad \text{and} \quad \delta_2 = \Delta \theta_1 + [(\Delta \theta)_0]_{\text{Exact}} - [(\Delta \theta)_0]_{3\text{rd order}}. \tag{4.2}
\]

where the expression for \((\psi_0)_{\text{Exact}}\) is given by Eq. (3.3a), whereas the expression for \((\psi_0)_{3\text{rd order}}\) can be found in [4]. A close examination of these two expressions for \((\psi_0)_{\text{Exact}}\) and \((\psi_0)_{3\text{rd order}}\) reveals that \(\delta_1\) is a function of the altitude \(\Phi\) and has nothing to do with the azimuth \(\Theta\). The same conclusion can be drawn for \(\delta_2\). Taking advantage of this finding, we may assume the azimuthal angle \(\Theta = 0\) for simplicity and our attention is concentrated to the evaluation of \((\delta_1, \delta_2)\) in the range of \((0, \Phi_m)\). Under the condition of \(\Theta = 0\), assume the target under tracking is located at the point \((x_1, y_1, -P)\) in the Cartesian coordinates system shown in Fig. 1, and then we have

\[
x_1 = P \tan \Phi \quad \text{and} \quad y_1 = 0. \tag{4.3}
\]

To evaluate the difference \((\Delta \theta)_{\text{Exact}} - (\Delta \theta)_{3\text{rd order}}\), we first substitute the target location specified by \((0, \Phi_m)\) into the third-order relation to determine prisms’ rotation angles \((\theta_1)_{3\text{rd order}}\) and \((\theta_2)_{3\text{rd order}}\), and then substitute from \((\theta_1)_{3\text{rd order}}\) and \((\theta_2)_{3\text{rd order}}\) into the expressions for exact solution presented in Section 3 of this study and then we obtain the point \((x_2, y_2, -P)\) where the laser beam hits the target plane located at \(z = -P\) if the two prisms are orientated according to the predictions of the third-order theory. The distance \(d\) between the two points \((x_1, y_1, -P)\) and \((x_2, y_2, -P)\) represents the position finding error in precision pointing the target that can be evaluated by using the formula

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{4.4}
\]

Because \((x_2, y_2, P)\) is proportional to direction cosines \([K_A, L_A, M_A]\) of the ray emergent from the system, we may obtain \((x_2, y_2)\) from the relationship

\[
\frac{x_2}{K_A} = \frac{y_2}{L_A} = \frac{-P}{M_A}. \tag{4.5}
\]
\[
\frac{d}{P} = \sqrt{\left(\frac{K_A}{M_A} + \tan \Phi\right)^2 + \left(\frac{L_A}{M_A}\right)^2},
\]

\[\theta_1 = (\theta_1)_{3rd\text{order}}, \quad \theta_2 = (\theta_2)_{3rd\text{order}}. \tag{4.6}\]

This equation is plotted in Fig. 4 under the condition of two identical prisms. The curves in Fig. 4 show the normalized position finding error \(d/P\) as a function of the normalized altitude \(\Phi/\Phi_m\) with the prism opening angle \(\alpha\) as a parameter. Specifically, Fig. 4(a) is for systems using glass prisms of \(n = 1.5\) and when the opening angle of the two prisms \(\alpha = 1', 2', \ldots, 10'\). It is seen from Fig. 4(a) if the position finding error \(d/P < 10^{-6}\) is required, then the third-order theory provides results with sufficient high accuracy only when the prism opening angle \(\alpha < 2'\) and the target should be located in the paraxial region of the scan field where \(\Phi \leq 0.2\Phi_m\). Similar situations can be seen in Fig. 4(b), which is plotted under the conditions of systems using silicon prisms of \(n = 4.0\) with opening angles of \(\alpha = 1', 2', \ldots, 5'\).

Finally, let us see a case example: assuming that a laser beam is required to be steered into the direction specified by the altitude \(\Phi = 45°\) and azimuth \(\Theta = 120°\). The prism system has two identical wedge prisms of index \(n = 4.0\) and opening angle \(\alpha = 5°\). If the two wedges are in the 21-12 arrangement, the first step is to obtain from Eq. (3.2a) the relative azimuth and \(\Phi_0 = 768°\), and then we can have the azimuth discrepancy \(\delta_\Pi\) and the prism rotation angles \(\Theta_\Pi\) as functions of the optical axis of the system.

The surprising outcome because under the condition of \(\alpha \leq 5°\) the third-order theory predicts results with high accuracy as discussed in Section 3. The surprising issue is that a small angular error may cause a huge position finding error if the target is located several hundred miles away, e.g., at 500 km the position error will be \(\sim 125\) m, which may be much larger than the maximum linear dimension of the target.

From this case example we may conclude that \(\delta_1\) and \(\delta_2\) in Eq. (4.2) relating to the differences between the predictions of different theories are small quantities. Therefore, the graphs in Fig. 5 of [4] showing the prism rotation angles \((\theta_1, \theta_2)\) as functions of beam steering angles \((\Phi, \Theta)\) will be good approximations for the prediction of the exact solution.

5. Motion Equations of the Risley Prisms to Create a Desirable Scan Pattern

Create a desirable scan pattern on some plane perpendicular to the optical axis of the system is also known as synthesis of the scan pattern [10–12]. In this section, scan pattern synthesis is considered as a generalization of the inverse problem, i.e., from steering a laser beam to address a single point to create a desirable pattern on the plane of observation.

Assume the desirable scan pattern is described by the parametrization

\[x(\tau) = \rho(\tau) \cos(\varphi(\tau)), \quad y(\tau) = \rho(\tau) \sin(\varphi(\tau)), \quad \varphi(\tau) = \arctan\left(\frac{y(\tau)}{x(\tau)}\right) \tag{5.1}\]

where \(\rho(\tau)\) and \(\varphi(\tau)\) are given by

\[\rho(\tau) = \sqrt{x^2(\tau) + y^2(\tau)} \tag{5.2a}\]

and

\[\varphi(\tau) = \arctan\left(\frac{y(\tau)}{x(\tau)}\right). \tag{5.2b}\]

Similar to Eq. (4.5) in Section 4, the Cartesian coordinates \([x(\tau), y(\tau)]\) for the scan pattern can be expressed in terms of the direction cosines of the ray emergent from the system:

\[x(\tau) = -\frac{K_A}{M_A} P \quad \text{and} \quad y(\tau) = -\frac{L_A}{M_A} P, \tag{5.3}\]

where \(P\) is the distance from the Risley prisms to the plane of observation (see Fig. 1). Upon substituting from Eq. (5.3) into Eq. (5.2) and then making use of the fact that the square sum of direction cosines is 1, we arrive at

\[
\left(\frac{\rho}{P}\right)^2 = x^2(\tau) + y^2(\tau) = \frac{K_A^2 + L_A^2}{M_A^2} = \frac{K_A^2 + L_A^2}{1 - (K_A^2 + L_A^2)}. \tag{5.4}\]
Again, upon substituting from Eqs. (T1.1), (T1.3), and (T1.5) into Eqs. (2.5a) and (2.5b) for \( K_A \) and \( L_A \) and then into Eq. (5.4), we obtain, after some rearranging, the expression

\[
a_1^2 + a_3^2 \sin^2 \alpha_2 + 2a_1 a_3 \sin \alpha_2 \cos(\Delta \theta) = \frac{\rho^2(\tau)}{\rho^2 + \rho^2(\tau)},
\]

(5.5)

where \( \Delta \theta = \theta_2 - \theta_1 \) represents the difference of prisms' orientations and \( \Delta \theta \) is a function of the time-dependent parameter \( \tau \).

After a close examination of the expressions for the coefficients \( (a_1, a_2, a_3) \) in Table 1, we found \( a_1 \) and \( a_2 \) are independent from \( \Delta \theta \) but \( a_3 \) is a function of \( \cos \Delta \theta \) and their functional relationship can be expressed as

\[
\cos \Delta \theta = \frac{1}{a_1 \sin \alpha_2} \left( \frac{1 - n_2^2 - a_2^2}{2a_3} \right) + a_2 \cos \alpha_2.
\]

(5.6)

Upon substituting from Eq. (5.6) into Eq. (5.5), we obtain the following second-order equation:

\[
Aa_3^2 + Ba_3 + C = 0,
\]

(5.7)

solved for the formula

\[
a_3 = \frac{3 \pm \sqrt{9 - 4AC}}{2A}.
\]

where

\[
A = \cos^2 \alpha_2, \quad B = -2a_2 \cos \alpha_2 \quad \text{and} \quad C = -a_2^2 - 1 + n_2^2 + \frac{\rho^2(\tau)}{\rho^2 + \rho^2(\tau)}.
\]

Again, upon substituting from \( a_3 \) in Eq. (5.8) into Eq. (5.6), we obtain

\[
\Delta \theta = \arccos \left[ \frac{1}{a_1 \sin \alpha_2} \left( \frac{1 - n_2^2 - a_2^2}{2a_3} \right) + a_2 \cos \alpha_2 \right].
\]

(5.9)

To determine the rotation angles \( \theta_1(\tau) \) and \( \theta_2(\tau) \) for the two prisms, we return to Eqs. (5.2b) and (5.3) and re-express Eq. (5.2b) in the form

\[
\tan \varphi(\tau) = \frac{y(\tau)}{x(\tau)} = \frac{L_A}{K_A} = \frac{a_1 \sin \theta_1 + a_3 \sin \alpha_2 \sin \theta_2}{a_1 \cos \theta_1 + a_3 \sin \alpha_2 \cos \theta_2}.
\]

(5.10)

On account of \( \Delta \theta = \theta_2 - \theta_1 \), Eq. (5.10) can be rearranged in the form

\[
\tan \varphi(\tau) = \frac{a_1 \sin \theta_1 + a_3 \sin \alpha_2 \sin (\theta_1 + \Delta \theta)}{a_1 \cos \theta_1 + a_3 \sin \alpha_2 \cos (\theta_1 + \Delta \theta)} = \frac{\sin(\theta_1 + \theta_{10})}{\cos(\theta_1 + \theta_{10})} = \tan(\theta_1 + \theta_{10}),
\]

(5.11)

where

\[
\theta_{10} = \arctan \left( \frac{a_3 \sin \alpha_2 \sin \Delta \theta}{a_1 + a_3 \sin \alpha_2 \cos \Delta \theta} \right).
\]

To this stage, we have sufficient parameters to express the rotation angles \( \theta_1 = \theta_1(\tau) \) and \( \theta_2 = \theta_2(\tau) \) of the two prisms in the form

\[
\theta_1 = \varphi(\tau) - \theta_{10} \quad \text{and} \quad \theta_2 = \varphi(\tau) - \theta_{10} + \Delta \theta.
\]

(5.12)

Finally, let us see a case example regarding the creation of an elliptical scan pattern [see Fig. 5(a)] on the plane of observation located at a distance \( P \) from the Risley prisms system using two identical wedges of \( n_1 = n_2 = n = 1.5 \) and \( a_1 = a_2 = a = 5^\circ \). The pattern to be created can be expressed parametrically in the form

\[
x = w_x \cos \tau \quad \text{and} \quad y = w_y \sin \tau,
\]

(5.13)

where \( w_x \) and \( w_y \) are the major and minor axes coinciding with the \( x \) and \( y \) axes and \( \tau \) is the time-dependent parameter running from 0 to \( 2\pi \).
Under the condition of $w_x = 0.08P$ and $w_y = 0.04P$, the azimuthal rotation angle $\Delta \theta$ between $\Pi_1$ and $\Pi_2$ is plotted from Eq. (5.9) and shown by the curve in Fig. 5(b), from which we found two extreme values $(\Delta \theta)_1 = 49.1^\circ$ and $(\Delta \theta)_2 = 129.6^\circ$. The two curves in Fig. 5(c) are plotted from Eq. (5.11) to show the dependence of prism rotation angles $\theta_1$ and $\theta_2$ on the time-dependent parameter $\tau$. The spacing attains the minimum value of $\pi$ when $\tau = 0$ to $2\pi$. The spacing attains the minimum value of $(\Delta \theta)_1 = 49.1^\circ$ when $\tau = 0$ and $\pi$, where the two prism apexes are aligned the prism system attains the maximum power of ray deviation for generation of the major axis of the ellipse, whereas another extreme value $(\Delta \theta)_2 = 129.6^\circ$ is reached when $\tau = 0.5\pi$ and $1.5\pi$... that implies the power of ray deviation of the system attains minima for generation of the minor axis of the ellipse.

Correctness of the above discussion can be checked by using the first-order theory for rotating prisms [1,3,4], which states that the resultant deviation of the light ray is the vector sum of the deviations contributed by the two prisms as shown in Fig. 1. For example, when $\tau = 0$ laser spot is at the point $(x = w_x, y = 0)$, the orientations of the prisms $\Pi_1$ and $\Pi_2$ are at $\theta_1 = -(\Delta \theta)_1/2 = -24.6^\circ$ and $\theta_2 = \pi/2 = 24.6^\circ$, respectively. The resultant deviation at the point is proportional to $\cos(\Delta \theta)/2$. The resultant deviation is proportional to the minor axis of the ellipse, the deviation is $\cos(\Delta \theta)/2 = \cos(24.6^\circ) = 0.9092$, $\cos(\Delta \theta)/2 = \cos(62.8^\circ) = 0.4571 \approx 2$

should take the value close to the ellipticity of the elliptical scan pattern, i.e., $\epsilon = w_x/w_y = 2$.

In this section, we have shown the generation of an elliptical scan pattern by a pair of identical and co-rotational prisms when their motions are under the control of the predictions of Eq. (5.12). However, generation of an elliptical scan pattern has been considered in existing literature [3] as a special case of optical generation of the hypocycloid by using a pair of counter-rotational prisms with different power in ray deviation.

6. Concluding Remarks

In conclusion, nonparaxial ray tracing through the Risley-prism-based systems in four different configurations has been performed under the conditions of the two prisms having different refractive index and opening angle. It is found that the four different configurations can be divided into two groups and a unified approach is developed that allows the use of two mathematical models to describe the four systems in different configurations. Closed form analytic solutions of the inverse problem, including both precision pointing and scan pattern synthesis, were developed and results obtained are compared with the predictions of the third-order theory and we found that even if the angular difference between the predictions of different theories are very small, the corresponding position error in tracking the target may not be small and could be much larger than the maximum linear dimension of the target.

References