LQR Controller for Stabilization of Flapping Wing MAVs in Gust Environments

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This study presents an approach to develop a controller for stabilization of a flapping wing micro-air vehicle (MAV) operating in gusty environments. The rigid-wing MAV is modeled as a nonlinear periodic system and the periodic-shooting method is used to find a trimmed periodic orbit. A linearized discrete-time representation of the system is created about this trimmed periodic orbit. This linearized representation is used for control synthesis based on linear quadratic regulator (LQR) theory. The kinematic variables defining the wing motion are used as control inputs. The controller is implemented on the nonlinear system model to stabilize the system in the presence of external disturbances, modeled as discrete gusts in this study. The performance of the controller, in terms of the gust speed tolerance of the nonlinear, closed-loop system, is compared for various controller designs. The LQR based controller is capable of stabilizing the MAV under longitudinal and lateral gust disturbances, however the maximum gust speed that can be tolerated by a given controller is influenced by a range of parameters, as discussed in the paper. Numerical simulations show that tolerance to longitudinal gusts is far greater than lateral tolerance. The study also shows that lateral control of the MAV can be achieved using only the wing-stroke magnitude and wing-stroke offset as the control inputs for each wing.

Nomenclature

ω Angular velocity (rad/sec)
Φ Vector of Euler angles (rad)
δ Reduction in wing-stroke frequency during first half of stroke-cycle (rad/sec)
η Wing feathering angle (rad)
ω Wing-stroke frequency (rad/sec)
φ Wing stroke angle (rad)
ρ Scaling factor for control-cost in LQR synthesis
θ Wing deviation angle (rad)

ω Wing-stroke frequency during second half of stroke-cycle (rad/sec)
̃ω Skew-symmetric angular rotation operator
ξ Phase-shift with frequency ̃ω during second half of stroke-cycle (rad)
b Superscript, indicates quantity defined for the body
w Superscript, indicates quantity defined for the wing
0 Subscript, indicates offset of wing motion dofs (rad), eg. φ0, θ0, η0
I. Introduction

Recent years have seen a significant amount of research in order to better understand and enhance the performance of flapping wing micro-air vehicles (MAVs). These studies have collectively established that multidisciplinary interactions can significantly influence the performance of such vehicles.\(^1,2\) This realization has important implications for design of flapping wing MAVs. Significant multidisciplinary interactions imply that a multidisciplinary design optimization (MDO) procedure be used, requiring models that adequately capture the cross-disciplinary couplings. Several recent studies have addressed cross-disciplinary coupling in flapping wing MAV models,\(^3,4,5,6,7,8\) with a view toward MDO.

An important requirement, often ignored in design studies, is the ability of the vehicle to withstand external disturbances, such as gusts. In conceptual design of conventional airplanes, little consideration is given to control authority and disturbance rejection, since these are not the primary design drivers for larger aircraft. For smaller airplanes, however, the sensitivity of the vehicle to external disturbances increases, and the ability to withstand them becomes more critical. Gust tolerance is far more important for flapping wing MAVs as they are inherently open-loop unstable during hovering flight and frequently operate under strict energy budgets. And while a typical gust encountered by a large airplane is a small fraction of the flight speed, a gust encountered by a MAV may be as fast as the vehicle. Hence, it is important to be able to design a vehicle with optimum mission performance under external disturbances.

The review article by Orlowski and Girard\(^9\) notes a general consensus among stability studies of flapping wing MAVs: in the absence of active control, the MAV system representations are unstable. Although they did not qualify this statement for a flight condition (hover, forward flight, climbing flight, etc.), our studies have found both forward flight and hover to be unstable. Of relevance here, is the work by Stanford et al.\(^10\)
where open-loop stability was considered for optimization of flapping wing kinematics. The authors were able to obtain open-loop stable designs, but concluded that closed-loop control is still needed for station-keeping in the presence of a disturbance, or for maintaining stability in the presence of large disturbances. Another important consideration is that the flight vehicle being studied is tailless, requiring that all control forces and moments be generated by variations in wing kinematics, according to some appropriate control law.

With regard to dynamic modeling of flapping wing MAVs Orlowski and Girard,\textsuperscript{9} and several of the references which they cite, highlight the importance of wing-mass inertia. The presence of flapping wings can exert significant inertial forces on the flight vehicle to which the wings are attached. These inertial forces influence the dynamics of a vehicle that is free to translate and rotate in space. The motion of the body, in turn, influences the wing motion and the resulting aerodynamic forces. This coupled flight dynamic behavior is nonlinear and time-varying for the system under consideration and can have a significant influence on the overall system performance. In the present study, we model the flight dynamic system including the wing-mass inertia for the purpose of closed-loop control. Recent derivations of nonlinear flight dynamic models for flapping wing MAVs can be found in papers by Sun, Wang and Xiong\textsuperscript{11} and Orlowski and Girard\textsuperscript{12}.

Orlowski and Girard\textsuperscript{9} note that none of the existing work on control of MAVs integrates a 6-degrees-of-freedom (6DOF) flight dynamic model that includes wing-mass inertia effects. Some of the recent work on control of this class of vehicles includes that of Khan and Agrawal,\textsuperscript{13} who used a very simple time-averaged model to study the implementation of a nonlinear controller for longitudinal flight. Doman, Oppenheimer and Sigthorsson\textsuperscript{14} used analytical expressions of cycle-average force and moments for 6DOF control of a flapping wing MAV. They later extended this work to obtain 6DOF control using wing-beat bias, as opposed to a bob-weight that had been included in their previous work.\textsuperscript{15} In their work, however, the system flight dynamics and wing inertial influence were neglected. More references on control of MAVs are available in the paper by Orlowski and Girard.\textsuperscript{9} A more recent work not found in this review article is that of Dietl and Garcia.\textsuperscript{16, 17} They studied the dynamics of longitudinal flight of an ornithopter with a fuselage and a horizontal and vertical tail. They focused on development of control laws to stabilize forward flight\textsuperscript{16} and later extended this work to enable transition from forward flight to hovering.\textsuperscript{17}

This study focuses on the design of controller for a flapping wing MAV with rigid wings, based on work by Orlowski and Girard,\textsuperscript{9} who used analytical expressions of cycle-average force and moments for 6DOF control of a flapping wing MAV. The flight dynamic equations are linearized about this trim motion and are used to construct a model-based controller using linear quadratic regulator (LQR) theory. This controller is then implemented in a numerical simulation of the nonlinear model in order to study the closed-loop system's response to gust disturbances. The emphasis is on understanding the influence of kinematic parameters used for control and on the performance of the controller under disturbances of increasing magnitude.

The wing kinematic description is discussed in Section II. The details of aerodynamic modeling and flight dynamics are discussed in Section III. The details of linearization and control synthesis are discussed in Section IV. The results from numerical studies are discussed in Section V. Finally, the conclusions from this study are discussed in the Section VI.

### II. Kinematic Parameterization

#### A. Parametric Definition

Here, we develop a kinematic parameterization that incorporates the split-cycle approach proposed by Doman, Oppenheimer and Sigthorsson\textsuperscript{15} and the earlier parameterization by Berman and Wang.\textsuperscript{18} This generalized parameterization allows one to smoothly vary the shape of a flapping cycle between two very different characteristic motions: triangular versus sinusoidal stroke-plane motion and square versus sinusoidal feathering motion. The parameterization is developed with a requirement to maintain continuity in wing angular motion, which is consistent with any physically realizable motion. The flexibility of parametrization developed here allows for several choices of control inputs, however, the control study presented in this paper does not make use of split-cycle control.

Doman, Oppenheimer and Sigthorsson\textsuperscript{15} proposed the use of three independent kinematic parameters for the wing: the split-cycle control parameter ($\delta$), the wing-beat bias ($\phi_0^W$) and the flapping frequency ($\omega$). The split-cycle control parameter changes the flapping frequency during the up- and down-strokes while keeping the overall flapping period constant at $2\pi/\omega$. The wing-beat bias shifts the flapping angle for an entire cycle
Our formulation includes a phase shift in the deviation ($\theta^W_0$) and feathering ($\eta^W_0$) angles. We assume, however, that split-cycle control does not change these values from the baseline kinematics. The control parameters include frequencies ($\omega$ and $\delta$), amplitudes ($\phi^W_m$, $\theta^W_m$ and $\eta^W_m$) and offsets ($\phi^W_0$, $\theta^W_0$ and $\eta^W_0$). In a manner consistent with split-cycle control, these are defined at the beginning of each cycle and remain invariant during the flapping period.

Kinematic parameters are defined independently for the left and right wings. Continuity of flapping angles across consecutive flapping cycles is ensured for each angle by modifying the first quarter of each flapping cycle. This is in contrast to the work by Doman, Oppenheimer and Sigthorsson, who applied the bias correction after a delay of one cycle. In their formulation, the control parameters for the $i^{th}$ cycle are $\delta_i$, $\omega_i$ and $\phi^W_{0,i-1}$ (as opposed to $\phi^W_{0,i}$). In the present work, a factor $A_\phi$ is calculated using $\phi^W_{0,i-1}$ and $\phi^W_{0,i}$ and is applied during the first one-fourth of the flapping cycle to ensure continuity. The calculation of the time derivatives of flapping angles assumes that the kinematic parameters are independent of time within a flapping cycle.

$K_\phi$ and $K_\eta$ provide control of stroke and feathering motion waveforms, respectively. A low value of both parameters ($\ll 1$) creates a sinusoidal wave, while $K_\phi = 1$ creates a triangular wave and $K_\eta > 1$ tends towards a square wave. The parameter $N_\theta$ is a multiplier of wing flapping frequency to obtain the wing deviation motion frequency, where a value of 1 gives an elliptical wing-tip motion, and a value of 2 gives a figure-of-eight wing tip motion.

The wing has three degrees of freedom: the azimuthal stroke plane angle ($\phi$), the wing-deviation angle ($\theta$) and the feathering, or wing-rotation angle ($\eta$). Before stating the kinematic parameterization, we define four non-dimensional parameters:

$$A_\phi = \frac{\phi^W_{m,old} + \phi^W_{0,old} - \phi^W_0}{\phi^W_m} - 1$$  
$$A_\theta = \frac{\theta^W_m \cos \theta^W_s + \theta^W_{0,old}}{\theta^W_m + 1} - 1$$  
$$\omega = \frac{\omega(t) + \pi}{\omega - 2\delta}$$  
$$\frac{\pi}{\omega - \delta} \leq t < \frac{2\pi}{\omega}$$

Here, the subscript old indicates a value from the previous flapping cycle. With these parameters defined, we present the following kinematic parameterization for flapping wing motion.

$$\phi^W(t) = \phi^W_m (1 + A_\phi) \frac{\sin^{-1}(K_\phi \sin((\omega - \delta)t + \frac{\pi}{2}))}{\sin^{-1} K_\phi} + \phi^W_0$$  
$$0 \leq t < \frac{\pi}{2(\omega - \delta)}$$

$$\phi^W(t) = \phi^W_m \frac{\sin^{-1}(K_\phi \sin((\omega - \delta)t + \frac{\pi}{2}))}{\sin^{-1} K_\phi} + \phi^W_0$$  
$$\frac{\pi}{2(\omega - \delta)} \leq t < \frac{\pi}{\omega - \delta}$$

$$\phi^W(t) = \phi^W_m \frac{\sin^{-1}(K_\phi \sin(\dot{\omega}t + \xi + \frac{\pi}{2}))}{\sin^{-1} K_\phi} + \phi^W_0$$  
$$\frac{\pi}{\omega - \delta} \leq t < \frac{2\pi}{\omega}$$

$$\theta^W(t) = (1 + A_\theta \cos((\omega - \delta)t)) (\theta^W_m \cos(N_\theta(\omega - \delta)t + \theta^W_s) + \theta^W_0)$$  
$$0 \leq t < \frac{\pi}{2(\omega - \delta)}$$

$$\theta^W(t) = \theta^W_m \cos(N_\theta(\omega - \delta)t + \theta^W_s) + \theta^W_0$$  
$$\frac{\pi}{2(\omega - \delta)} \leq t < \frac{\pi}{\omega - \delta}$$

$$\theta^W(t) = \theta^W_m \cos(N_\theta \dot{\omega}t + N_\theta \xi + \theta^W_s) + \theta^W_0$$  
$$\frac{\pi}{\omega - \delta} \leq t < \frac{2\pi}{\omega}$$
\[ \eta^W(t) = \left( \eta^W_m \frac{(\omega - \delta)t}{\pi/2} + \eta^W_{m,old} \left(1 - \frac{(\omega - \delta)t}{\pi/2}\right) \right) \tanh(K_\eta \sin((\omega - \delta)t + \eta^W_s)) \tan(K_\eta) \]
\[ + \left( \eta^W_0 \frac{(\omega - \delta)t}{\pi/2} + \eta^W_{0,old} \left(1 - \frac{(\omega - \delta)t}{\pi/2}\right) \right) 0 \leq t < \frac{\pi}{2(\omega - \delta)} \]  
\[ \eta^W(t) = \eta^W_m \frac{\tanh(K_\eta \sin((\omega - \delta)t + \eta^W_s))}{\tan(K_\eta)} + \eta^W_0 \quad \frac{\pi}{2(\omega - \delta)} \leq t < \frac{\pi}{\omega - \delta} \]  
\[ \eta^W(t) = \eta^W_m \frac{\tanh(K_\eta \sin(\tilde{\omega}t + \xi + \eta^W_s))}{\tan(K_\eta)} + \eta^W_0 \quad \frac{\pi}{\omega - \delta} \leq t < \frac{2\pi}{\omega} \]

An example of wing kinematic description with varying split-cycle control and wing-beat across consecutive flapping cycles is provided in Table 1. The wing angles obtained from the kinematic parameterization presented in this section are plotted in Fig. 1. Note that \( C^0 \) continuity is maintained across consecutive cycles for each case.

### Table 1. Kinematic parameters describing wing motion with varying wing-beat bias and split-cycle frequency across two consecutive cycles

<table>
<thead>
<tr>
<th>Param</th>
<th>Val</th>
<th>Param</th>
<th>Val</th>
<th>Param</th>
<th>Val</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega )</td>
<td>125</td>
<td>( N_\theta )</td>
<td>2</td>
<td>( K_\eta )</td>
<td>4</td>
</tr>
<tr>
<td>( \delta )</td>
<td>30</td>
<td>( \theta^W_m )</td>
<td>1</td>
<td>( \eta^W_m )</td>
<td>1</td>
</tr>
<tr>
<td>( K_\phi )</td>
<td>0.001</td>
<td>( \theta^W_0 )</td>
<td>0.2</td>
<td>( \eta^W_0 )</td>
<td>0.2</td>
</tr>
<tr>
<td>( \phi^W_m )</td>
<td>1</td>
<td>( \theta^W_s )</td>
<td>0.785</td>
<td>( \eta^W_s )</td>
<td>-0.785</td>
</tr>
<tr>
<td>( \phi^W_0 )</td>
<td>0.2</td>
<td>( \theta^W_{m,old} )</td>
<td>0.75</td>
<td>( \eta^W_{m,old} )</td>
<td>0.75</td>
</tr>
<tr>
<td>( \phi^W_s )</td>
<td>0.7</td>
<td>( \theta^W_{0,old} )</td>
<td>0.2</td>
<td>( \eta^W_{0,old} )</td>
<td>-0.2</td>
</tr>
<tr>
<td>( \phi^W )</td>
<td>-0.1</td>
<td>( \theta^W_{s,old} )</td>
<td>0.2</td>
<td>( \eta^W_{s,old} )</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

### Figure 1. Wing angles with split-cycle frequency, wing-beat bias and variation of flapping magnitude and bias between two cycles

#### B. A Note on Terminology

In order to facilitate discussion in later sections, it is clarified that a set of kinematic parameters is represented as \( q \). The complete definition of wing kinematic motion during a given cycle is defined with a total of 12 parameters: \( \omega, \delta, \phi^W_m, \phi^W_0, K_\phi, \theta^W_m, \theta^W_0, \theta^W_s, \eta^W_m, \eta^W_0, \eta^W_s \) and \( K^W_\eta \). The numerical procedure to obtain
a trim orbit (discussed later in Sec. IV.A) uses two kinematic parameters, and this subset is referred to as \(q_{\text{trim}}\). The numerical values of \(q_{\text{trim}}\) are known for a trimmed vehicle. For such a vehicle the control synthesis procedure (discussed in Sec. IV.C) defines the set of kinematic variables, \(q_{\text{control}}\), that will be used by a given controller. This set of kinematic variables can contain one or all of the kinematic parameters listed above, and is independent of the kinematic parameters used for trim.

III. Flight Dynamics

The system under consideration for this work is that of a rigid tailless body connected to a series of rigid flapping wings (two for this work, though this is not restricted in the derivation). The wings are pinned to the body. The orientation of each wing with respect to the body is defined by the three Euler angles \(\phi^W\), \(\theta^W\), and \(\eta^W\) corresponding to that particular wing. For clarity and economy of notation, we make no distinction between the port and starboard wing in this presentation. The wing motion (i.e., the time histories of \(\phi^W\), \(\theta^W\), and \(\eta^W\)) is prescribed, but the overall motion of the body is computed using the flight dynamics model developed below.

A. Base Body Kinematics and Dynamics

The system under consideration is depicted in Fig. 2. There exists a fixed inertial coordinate system \((x_I, y_I, z_I)\), a coordinate system attached to the center of gravity of the fuselage body \((x_B, y_B, z_B)\), located at point B, and a third coordinate system \((x_W, y_W, z_W)\) located at the hinge point H, which rotates with the rigid wing. The center of gravity of the wing is located at point W. The attitude and position of the body is defined by

\[
\Phi^B = \begin{bmatrix}
\phi^B \\
\theta^B \\
\eta^B 
\end{bmatrix}
\]

(14)

\[
r^I_B = \begin{bmatrix}
x_I \\
y_I 
\end{bmatrix}
\]

(15)

The angular velocity of the body, written in the body frame, is

\[
\omega^B = \begin{bmatrix}
\cos \theta^B \cos \eta^B & \sin \eta^B & 0 \\
-\cos \theta^B \sin \eta^B & \cos \eta^B & 0 \\
\sin \theta^B & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}^B \\
\dot{\theta}^B \\
\dot{\eta}^B
\end{bmatrix}
= E_{BI} \dot{\Phi}^B
\]

(16)

The transformation matrix from the body frame to the inertial frame is

\[
T_{BI} = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \phi^B & -\sin \phi^B \\
0 & \sin \phi^B & \cos \phi^B
\end{bmatrix}
\begin{bmatrix}
\cos \theta^B & 0 & \sin \theta^B \\
0 & 1 & 0 \\
-\sin \theta^B & 0 & \cos \theta^B
\end{bmatrix}
\begin{bmatrix}
\cos \eta^B & -\sin \eta^B & 0 \\
\sin \eta^B & \cos \eta^B & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(17)

The velocity of the body is

\[
\dot{r}^B_I = \begin{bmatrix}
\dot{x}^B_I \\
\dot{y}^B_I \\
\dot{z}^B_I
\end{bmatrix} = v^B_I = T_{BI} v^B_B
\]

(18)

where \(v^B_B\) is the velocity of point B written in the body-attached coordinate system. The acceleration of the body is

\[
\ddot{v}^B_I = T_{BI} \ddot{v}^B_B + \dot{T}_{BI} v^B_B
\]

(19)

where the tilde indicates the skew-symmetric matrix satisfying \(\tilde{a}b = a \times b\) for vectors \(a\) and \(b\).
The prescribed kinematics of the wing (i.e., the attitude of the wing with respect to the body) are governed by three Euler angles

\[
\Phi^W = \begin{Bmatrix}
\phi^W \\
\theta^W \\
\eta^W 
\end{Bmatrix}
\]  

(20)

The angular velocity of the wing with respect to the body is

\[
\omega^W = \begin{bmatrix}
-\sin \theta^W & 1 & \cos \eta^W & 0 \\
\cos \theta^W & 0 & -\sin \eta^W & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{\phi}^W \\
\dot{\theta}^W \\
\dot{\eta}^W
\end{bmatrix}
\]  

(21)

The transformation matrix from the wing frame to the body frame is

\[
T_{WB} = \begin{bmatrix}
\cos \phi^W & -\sin \phi^W & 0 \\
\sin \phi^W & \cos \phi^W & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \theta^W & 0 & \sin \theta^W \\
0 & 1 & 0 \\
-\sin \theta^W & 0 & \cos \theta^W
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \eta^W & -\sin \eta^W \\
0 & \sin \eta^W & \cos \eta^W
\end{bmatrix}
\]  

(22)

The equations of motion have been derived by Sun, Wang and Xiaong; only the final form is given here:

\[
\begin{bmatrix}
1 \\
I \\
I_{m_{tot}} \\
I_B
\end{bmatrix}
\begin{bmatrix}
\dot{r}_B^I \\
\dot{\Phi}_B^I \\
\dot{v}_B^I \\
\dot{\omega}_B^I
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
m_{tot} \omega_B^I \\
\bar{\omega}_B^I I_B
\end{bmatrix}
= \begin{bmatrix}
\dot{r}_B^E \\
\dot{\Phi}_B^E \\
\dot{v}_B^E \\
\dot{\omega}_B^E
\end{bmatrix}
+ \begin{bmatrix}
P_{\text{force}} \\
P_{\text{moment}}
\end{bmatrix}
+ \begin{bmatrix}
T_{BI} \omega_B^I \\
E_{BI}^1 \omega_B^I \\
P_{\text{force}} \\
P_{\text{moment}}
\end{bmatrix}
\]  

(23)

where \( I \) is the identity matrix, \( m_{tot} \) is the complete mass of the vehicle (body plus wings), and \( I_B \) is the inertia tensor of the body, written in the body coordinate system at point B. The vectors \( P_{\text{force}} \) and \( P_{\text{moment}} \) are inertial forces and moments imposed on the body due to the wing motion, while \( F_{\text{force}} \) and \( F_{\text{moment}} \) are gravitational and aerodynamic terms.

### B. Aerodynamics

The aerodynamic terms are computed with the two-dimensional quasi-steady blade element model discussed by Berman and Wang and used in the subsequent study by Stanford et al. The wing is discretized into
a series of blade elements for which two forces (lift and drag) and a pitching moment are calculated based on quasi-steady assumptions.

It should be noted that this aerodynamic model cannot capture many potentially important physical phenomena: aerodynamic interactions between the two wings, interactions between the wings and the body, flow over the body itself, or unsteady inflow from the shed wake. Despite these shortcomings, favorable comparisons of this model with higher-fidelity Navier-Stokes solvers have been demonstrated for a flapping wing pinned to a stationary point.\textsuperscript{18}

IV. Trim, Linearization and Control Synthesis

A. Trim

The physical system under consideration is modeled by a system of nonlinear, periodic equations. In order to proceed with control synthesis, the system is linearized about a trim state: a periodic orbit of the dynamic equations with the same period as the wing flapping cycle.

Following the approach developed by Stanford, et al.,\textsuperscript{10} a periodic-shooting method is used to obtain trim states for this system. The procedure defines two kinematic parameters \( q_{\text{trim}} \) and the body pitch-angle and angular-rate at the beginning of a cycle as the unknowns. A Newton iteration is used to calculate the solution, which requires sensitivity of the states with respect to the kinematic parameters.

B. Linearized Representation

The present work uses linear quadratic regulator (LQR) theory for the synthesis of a controller to stabilize the system in the presence of disturbances. The system state at time \( t_0 \) is defined by the linear and angular position and their time derivatives:

\[
\begin{bmatrix} x(t_0) \end{bmatrix} = \begin{bmatrix} r^B_I \\ \Phi^B_I \\ v^B_I \\ \omega^B_I \end{bmatrix}
\]

Having numerically determined a periodic trim state, the system is linearized about this periodic orbit by calculating the sensitivity of the end-of-period states with respect to change in the states and kinematic parameters at the beginning of the period. For the purpose of this study it is assumed that the state perturbations can be calculated and are available to the controller.

If the flight condition is trimmed, and no disturbances are present, the system will follow the periodic orbit during one flapping cycle and the state at the end of this cycle will be identical to the initial state: \( x^{T+t_0} = x^t_0 \). This will not be the case, however, if the initial state is perturbed by a small amount \( dx^{t_0} \). Depending on the nature of the system, the result may be a growing (unstable) or decaying (stable) discrepancy between the true state and the trimmed state. In any case, however, an initial perturbation will result in some nonzero perturbation at the end of the period, represented as \( dx^{T+t_0} \).

The nonlinear system shown in Eq. (23) is now represented in a generic form as (25).

\[
\begin{align*}
\dot{x} &= R(x, \dot{x}, q) \\
x(0) &= x^t_0
\end{align*}
\]  

(25)

(26)

In order to calculate \( dx^{T+t_0} \), given the initial disturbance \( dx^{t_0} \), we compute the sensitivity of the system equations to small changes in the initial state:

\[
\begin{bmatrix} \frac{dx}{dx^{t_0}} \\ \frac{dx(0)}{dx^{t_0}} \end{bmatrix} = \begin{bmatrix} \frac{\partial R}{\partial x} \\ \frac{\partial R}{\partial \dot{x}} \end{bmatrix} \begin{bmatrix} dx \\ dx^{t_0} \end{bmatrix} + \begin{bmatrix} \frac{\partial R}{\partial x} \\ \frac{\partial R}{\partial \dot{x}} \end{bmatrix} \begin{bmatrix} dx \\ dx^{t_0} \end{bmatrix}
\]

\[
\frac{dx(0)}{dx^{t_0}} = I_{12 \times 12}
\]

(27)

(28)

The system (27) is a linear ordinary matrix differential equation with \( \frac{dx}{dx^{t_0}} \) as the unknown. The quantities \( \frac{\partial R}{\partial x} \) and \( \frac{\partial R}{\partial \dot{x}} \) are available from the solution of Eq. (25). Integrating Eq. (27) in time for one period, one
obtains the sensitivity of the state at the end of one flapping cycle \((t = t_0 + T)\) with respect to the state at the beginning of the flapping cycle \((t = t_0)\). This sensitivity matrix may then be used to compute

\[
dx^{T+t_0} = \left[ \frac{dx}{d\mathbf{q}^0} \right]_{t=T+t_0} dx_t^0
\] (29)

The sensitivity matrix in Eq. (29) is also called the \textit{monodromy matrix}\textsuperscript{19} or the \textit{state-transition matrix}\textsuperscript{20}. This matrix is calculated as a part of the periodic-shooting method used to obtain a trimmed solution. The reader is referred to the preceding work by Stanford et al.\textsuperscript{10} for more details on this process.

The evolution of the system state is affected by changes in initial conditions, as discussed above, changes in the wing kinematic variables, \(dq^0\) and \(dq^{t_0-T}\), and any other external disturbances, termed \(\Delta^{t_0}\). Although the controller will have access to the subset \(\mathbf{q}_{\text{control}}\), the formulation presented here is applicable to any subset of kinematic variables. Hence, to maintain generality, the vector of kinematic variables is represented as \(\mathbf{q}\). The sensitivities of \(\mathbf{x}^{T+t_0}\) with respect to the latter two parameters can be calculated using the same procedure that led us to Eq. (29). For perturbations due to the kinematic parameters, \(\mathbf{q}\), the equation is

\[
\begin{align*}
\frac{dx}{dq^0} & = \frac{\partial R}{\partial \mathbf{x}} \frac{dx}{dq^0} + \frac{\partial R}{\partial \mathbf{q}^0} \frac{dx}{dq^0} + \frac{\partial R}{\partial \mathbf{q}^0} \\
\frac{dx(0)}{dq^{t_0-T}} & = 0 \\
\frac{dx}{dq^{t_0-T}} & = \frac{\partial R}{\partial \mathbf{x}} \frac{dx}{dq^{t_0-T}} + \frac{\partial R}{\partial \mathbf{q}^0} \frac{dx}{dq^{t_0-T}} + \frac{\partial R}{\partial \mathbf{q}^0} \\
\frac{dx(0)}{dq^{t_0-T}} & = 0
\end{align*}
\] (30)

\[
\begin{align*}
\frac{dx^{T+t_0}}{dq^{t_0}} & = \left[ \frac{dx}{d\mathbf{q}^0} \right]_{t=T+t_0} dq^{t_0} + \left[ \frac{dx}{dq^{t_0-T}} \right]_{t=T+t_0} dq^{t_0-T} \\
\frac{dx(0)}{dq^{t_0-T}} & = 0
\end{align*}
\] (34)

\[
\frac{dx^{T+t_0}}{d\Delta^{t_0}} = \left[ \frac{dx}{d\Delta^{t_0}} \right]_{t=T+t_0} \Delta^{t_0} \] (37)

Eqs. (29), (34) and (37) can be combined to give

\[
dx^{T+t_0} = \left[ \frac{dx}{dx^0} \right]_{t=T+t_0} dx^0 + \left[ \frac{dx}{dq^0} \right]_{t=T+t_0} dq^0 + \left[ \frac{dx}{dq^{t_0-T}} \right]_{t=T+t_0} dq^{t_0-T} \\
+ \left[ \frac{dx}{d\Delta^{t_0}} \right]_{t=T+t_0} \Delta^{t_0}
\] (38)

In simpler notation, we have

\[
dx = A dx^0 + B dq^0 + B_{-1} dq^{-1} + D \Delta^0
\] (39)

where the superscripts represent the following short-hand for time stamps: \(0 = T, 1 = t_0 + T\) and \(-1 = t_0 - T\).
and where

\[
A = \begin{bmatrix}
\frac{dx}{dt} \\
\frac{dx}{dt} + B_0 \frac{dq}{dt}
\end{bmatrix}_{t=T+t_0} \tag{40}
\]

\[
B = \begin{bmatrix}
\frac{dx}{dt} \\
\frac{dq}{dt}
\end{bmatrix}_{t=T+t_0} \tag{41}
\]

\[
B_{-1} = \begin{bmatrix}
\frac{dx}{dt} \\
\frac{dq}{dt}
\end{bmatrix}_{t=T+t_0} \tag{42}
\]

\[
D = \begin{bmatrix}
\frac{dx}{dt} \\
\frac{d\Delta}{dt}
\end{bmatrix}_{t=T+t_0} \tag{43}
\]

Eq. (39) is a linear, discrete-time system which maps an initial state, at the beginning of a flapping cycle, to a final state at the end of the flapping cycle. As a linearization, this map is only approximate, but the approximation is quite good when the flapping period is small relative to the time constants of the base body’s motion.

The linear system (39) provides several opportunities with regard to the nonlinear, time-varying system (25). First, it provides a mechanism to calculate the change in system state due to a change in parameters, provided the parameter change is small, without need for solving the governing nonlinear equation. Second, the eigenvalues of \( A \) characterize the stability of the trim state. Eigenvalues with magnitude greater than one are associated with unstable modes; eigenvalues with magnitude less than one correspond to stable modes. Complex eigenvalues, which can occur only in conjugate pairs, are associated with oscillatory modes while real-valued eigenvalues are associated with non-oscillatory modes. Third, the linearized system lends itself to control synthesis using linear optimal control approaches, such as LQR synthesis.

C. LQR Control Synthesis

LQR theory provides a method of determining a linear, static state feedback control law \( d\tilde{x}^0 = -Kd\tilde{x} \) (44) for which the gain matrix \( K \) minimizes a quadratic cost function for a control system with linear dynamics.

To develop an LQR controller for the flapping wing MAV model described earlier, Eq. (39) is first converted to state-space form

\[
d\tilde{x}^1 = \tilde{A}d\tilde{x}^0 + \tilde{B}d\tilde{q}^0 + \tilde{D}\Delta^0 \tag{45}
\]

\[
d\tilde{x}^0 = \begin{cases}
d\tilde{x}^0 \\
d\tilde{q}^{-1}
\end{cases} \tag{46}
\]

\[
d\tilde{x}^1 = \begin{cases}
d\tilde{x}^1 \\
d\tilde{q}^0
\end{cases} \tag{47}
\]

\[
\tilde{A} = \begin{bmatrix}
A & B_{-1} \\
0 & 0
\end{bmatrix} \tag{48}
\]

\[
\tilde{B} = \begin{bmatrix}
B \\
B_{0\rightarrow -1}
\end{bmatrix} \tag{49}
\]

\[
\tilde{D} = \begin{bmatrix}
D \\
0
\end{bmatrix} \tag{50}
\]

In the state vector \( d\tilde{x}^0 \), the term \( d\tilde{q}^{-1} \) is the vector of kinematic parameters from the previous cycle that influence the state variables during the current cycle. These include the magnitude and bias of the three wing degrees-of-freedom. The matrix \( B_{0\rightarrow -1} \) is used to map the \( dq^0 \) to \( d\tilde{q}^0 \), and depends on the choice of kinematic parameters used for control. As an example, if the wing stroke magnitude \( (\phi_m) \), bias \( (\phi_0) \) and sharpness parameter \( (K_\phi) \), the wing deviation magnitude \( (\theta_m) \), and the wing feathering magnitude \( (\eta_m) \), etc.,